Practice problems–MC Integration and Importance Sampling

Problem 1

Estimate the following integral:

$$\int_0^\infty x^{3/2} e^{-x} dx.$$

a) Using classical Monte Carlo integration.
b) Using importance sampling with $\phi(x)$ the density of a chi-square distribution on 1 degree of freedom.
c) Using importance sampling with $\phi(x)$ the density of a chi-square distribution on 5 degrees of freedom.

Problem 2

Estimate the following integral:

$$\int_0^\pi \frac{dx}{x^2 + \cos^2 x}.$$

a) Using classical Monte Carlo integration.
b) Using importance sampling with $\phi(x)$ the density of an exponential distribution with mean 1.
c) **Bonus**: Consider a general exponential distribution with mean $1/\lambda$ for the importance function ϕ. For which value of λ can you minimize the variance of the estimate?

Problem 3

We will look at an application of Monte Carlo integration to financial statistics. Let V_0 be the original value of a portfolio, and let V_1 be its value after one unit of time (e.g. a day, a week, a month). Define the **loss** as follows:

$$L = -(V_1 - V_0).$$

In other words, if the value of the portfolio decreased, L will be a positive number representing the value lost.

We are interested in estimating the **Value-at-Risk**, which is defined as the real number x such that

$$P(L > x) = p,$$

for a probability p (e.g. $p = 0.01$). In other words, the probability of losing value of x or more is equal to p.

For simplicity, we will assume that V_0 is fixed and $V_1 \sim N(V_0, \sigma^2)$.

a) Derive the distribution of L.
b) Fix $V_0 = 1000$ and $\sigma = 100$ for the rest of this problem. Using the hit-or-miss approach, give an estimate of $P(L > 0)$, i.e. the probability that the loss is positive. Choose a sample size such that the length of an approximate 95% confidence interval is about 0.04.
c) Now fix $p = 0.05$. Find an estimate of the Value-at-Risk x. (**Hint**: You will probably have to use the hit-or-miss approach multiple times.)
d) Repeat parts b) and c) using Importance Sampling. Be explicit about your choice of importance function(s).

Problem 4

Suppose we want to estimate the expected value

$$E(\exp(-0.5(X-10)^2)),$$

where X is a standard normal, using importance sampling. You can choose between three different normal distributions (same variance, different mean) for the importance function $\phi(x)$:

- $N(0,1)$
- $N(5,1)$
- $N(10,1)$

a) Using a graphical approach, explain which candidate should lead to the smallest variance.
b) Confirm your guess by computing estimates using all three importance functions and providing standard error estimates.

Problem 5

Consider the following density function:

$$f(x) = \frac{2}{\pi(x^2+4)},$$

where x is any real number. You can sample from this distribution using the inverse-transform method.

Consider the following integral:

$$\theta = \int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+4} dx.$$

a) By sampling from the distribution above, use Monte Carlo integration to compute an estimate of θ.
b) Compute an estimate of the same integral using Importance Sampling and the Cauchy distribution.
c) Find another importance function and compute a third estimate of θ.