Practice problems—Numerical Methods

Problem 1

Recall that the secant method starts with two initial values $x_0 \neq x_1$ and constructs a sequence:

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

Show that the secant iterations can be redefined as

$$x_n = \frac{x_{n-1} f(x_{n-2}) - x_{n-2} f(x_{n-1})}{f(x_{n-2}) - f(x_{n-1})}.$$

Problem 2

This problem is adapted from Süli & Mayers *An Introduction to Numerical Analysis* (2003). They define a variant of the secant method as follows. Define two sequences u_n and v_n such that all the values $f(u_n)$, $n = 0, 1, 2, \ldots$, have one sign, and all the values $f(v_n)$, $n = 0, 1, 2, \ldots$, have the opposite sign. From these two sequences, we define

$$w_n = \frac{u_n f(v_n) - v_n f(u_n)}{f(v_n) - f(u_n)}, \qquad n = 0, 1, 2, \ldots$$

Finally, define $u_{n+1} = w_n, v_{n+1} = v_n$ if $f(w_n)$ has the same sign as $f(u_n)$, and otherwise define $u_{n+1} = u_n, vn + 1 = w_n$.

In other words, start with two initial values u_0, v_0 such that $f(u_0), f(v_0)$ have different signs, compute w_0 and update both sequences depending on the sign of $f(w_0)$ and make sure the sign constrain is satisfied for both sequences. Continue until convergence.

Implement this algorithm in `R`, and use your implementation to find the root of the function

$$f(x) = \exp(x) - x - 2.$$

You can use $u_0 = 0, v_0 = 2$.

Problem 3

In this problem, you will implement Brent's method. Recall the algorithm from the lecture.

Algorithm

Start with interval $[a, b]$ and continuous function $f(x)$. The values $f(a), f(b)$ have opposite signs.

1. Define a third point $(c, f(c))$, where c is the value at which a linear interpolation crosses the x-axis. Depending on the sign of $f(c)$, we know the solution $f(x) = 0$ falls inside the interval (a, c) or (c, b).

2. Fit a sideways parabola to all three points, and find the intersection x_1 with the x-axis. If x_1 falls outside the interval from Step 1, replace x_1 by the midpoint of the interval (i.e. bisection).
3. Repeat until convergence.

Answer the following questions.

a. Look at the help page for the function `poly_calc` in the package `PolynomF`. Explain how you can use it to find the *sideways* parabola passing through three points $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$.
b. Assume `poly_fun` is the output of `poly_calc` as used in part a. to fit a sideways parabola. Explain how you can use `poly_fun` to compute the value `x` at which the parabola crosses the x-axis.
c. Use the answers from the previous two parts to implement Brent's algorithm in `R`. Use your implementation to find the root of the function $f(x) = \exp(x) - x - 2$. You can use the interval $[0, 2]$.
d. Compare your answer with the one you get from using `uniroot` (both the root and the number of iterations). Are they the same?
e. Look at the Wikipedia page on Brent's algorithm: https://en.wikipedia.org/wiki/Brent%27s_method. Describe how the actual algorithm differs from the description above. This is why we use the implementation from `uniroot`!