Problem Set 3–STAT 7200

1. Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be a random sample with $\mathbf{Y}_i \sim N_p(0, \Sigma)$, and write $\mathbb{Y}$ for the $n \times p$ matrix whose i-th row is $\mathbf{Y}_i$. Let $C = \frac{1}{n}\mathbf{1}^T$, where $\mathbf{1}$ is the n-dimensional vector of ones, and let $A = I_n - \frac{1}{n}\mathbf{1}\mathbf{1}^T$. Show that
   (a) $\mathbb{Y}^T A \mathbb{Y} = (n-1)S_n$;
   (b) $C\mathbb{Y} = \bar{\mathbf{Y}}^T$.

2. Let $S \sim W_p(m, \Sigma)$, and let B be a $q \times p$ matrix. Show that
$$BSB^T \sim W_p(m, B\Sigma B^T).$$

3. Let $S \sim W_p(m)$, with $m \geq p$. Show that
   (a) $\frac{1}{\mathbf{t}^T S^{-1} \mathbf{t}} \sim \chi^2(m-p+1)$ for any $\mathbf{t} \in \mathbb{R}^p$ with unit norm.
   (b) If $\mathbf{Y}$ and S are independent and $\mathbf{Y} \neq 0$ almost surely, then $\mathbf{Y}$ is independent of
$$\frac{\mathbf{Y}^T\mathbf{Y}}{\mathbf{Y}^T S^{-1} \mathbf{Y}} \sim \chi^2(m-p+1).$$

   Hint: You can use the fact that if H is an orthogonal matrix, then $HSH^T \sim W_p(m)$.

4. Let $S \sim W_p(m)$ with $m \geq p$, and consider the correlation matrix R, where the (i,j)-th entry is given by
$$r_{ij} = \frac{w_{ij}}{w_{ii}^{1/2} w_{jj}^{1/2}}.$$
   Show that the density of R is given by
$$f(R) = \frac{(\Gamma(m/2))^p}{\Gamma_p(m/2)} |R|^{(m-p-1)/2}.$$

   Hint: Use the transformation $S \mapsto \big(w_{11}, \ldots, w_{pp}, R\big)$.

5. Let $S \sim W_p(m, \Sigma)$, and let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$ be fixed. Show that the quadratic forms $\mathbf{a}^T S\mathbf{a}$ and $\mathbf{b}^T S\mathbf{b}$ are independent if and only if $\mathbf{a}^T \Sigma\mathbf{b} = 0$.