Problem Set 5–STAT 7200

1. Consider the following mixed-effect ANOVA model:

$$Y_{ij} = \mu + \delta_i + \tau_j + \epsilon_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p,$$

where $\delta_i \sim N(0, \sigma_\delta^2)$ and $\epsilon_{ij} \sim N(0, \sigma^2)$ are all mutually independent. The parameters μ and τ_j are unknown, and they satisfy the constraint $\sum_{j=1}^{p} \tau_j = 0$. Let $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{ip})$ and show that

$$E(\mathbf{Y}_i) = (\mu + \beta_1, \mu + \beta_p),$$
$$\text{Cov}(\mathbf{Y}_i) = \sigma_\delta^2 \mathbf{1}_p \mathbf{1}_p^T + \sigma^2 I_p.$$

Moreover, show that $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ are independent and identically distributed $N_p(\boldsymbol{\eta}, \Sigma_s)$, where

$$\boldsymbol{\eta} = (\mu + \beta_1, \mu + \beta_p),$$
$$\Sigma_s = \sigma_\delta^2 \mathbf{1}_p \mathbf{1}_p^T + \sigma^2 I_p.$$

2. Using the same notation as Problem 1: let $H = (h_1, H_2)$ be a $p \times p$ matrix whose first column is h_1 is $\frac{1}{\sqrt{p}} \mathbf{1}_p$. Consider the transformation

$$\mathbf{X}_i = H_2^T \mathbf{Y}_i, \quad i = 1, \ldots, n.$$

Show that $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are independent and normally distributed $N_p(0, \boldsymbol{\xi}, \sigma^2 I_{p-1})$, where $\boldsymbol{\xi} = H_2^T \boldsymbol{\eta}$. Use this property to construct a test for $H_0 : \Sigma = \Sigma_s$.

3. Given a random sample $\mathbf{Y}_i \sim N_p(\mu, \Sigma)$, $i = 1, \ldots, n$, with Σ positive definite, show that the likelihood ratio test for

$$H_0 : \Sigma = \gamma \Sigma_0,$$

where Σ_0 is known, is given by

$$\Lambda^{2/n} = \frac{|\Sigma_0^{-1} V|}{\left(\frac{1}{p} \text{tr} \Sigma_0^{-1} V\right)^p}.$$

Give an approximation to its distribution under H_0.

4. Let $\mathbf{Y}_{\ell 1}, \ldots, \mathbf{Y}_{\ell n_\ell} \sim N_p(\mu_\ell, \Sigma)$, with $\ell = 1, \ldots, g$. Suppose that we decompose $\mathbf{Y}_{\ell i} = (\mathbf{Y}^1_{\ell i}, \mathbf{Y}^2_{\ell i})$. Accordingly, we can decompose

$$\mu_\ell = (\mu_{1\ell}, \mu_{2\ell}), \quad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T = W + B$.
Assume that the second components of the means are equal, i.e. $\mu_{21} = \cdots = \mu_{2g}$. Consider the hypothesis test where H_0 is that the *first* components of the means are equal, i.e. $H_0 : \mu_{11} = \cdots = \mu_{1g}$.[1] Show that the likelihood ratio Λ is given by

$$\Lambda^{2/n} = \frac{|W_{11|2}|}{|T_{11|2}|} = \frac{|W|}{|T|} \left(\frac{|W_{11}|}{|T_{11}|} \right)^{-1},$$

where $W_{11|2} = W_{11} - W_{12} W_{22}^{-1} W_{21}$ and similarly for $T_{11|2}$.

5. Let $\lambda_1, \ldots, \lambda_s$ be the eigenvalues of the matrix $W^{-1}B$, where $s = \min(p, g-1)$ and where W, B are the within (and between) sum of squares and cross-products from MANOVA. Show that the following equalities hold:

$$\frac{|W|}{|B+W|} = \prod_{i=1}^{s} \frac{1}{1+\lambda_i}$$
$$\operatorname{tr}\left(B(B+W)^{-1}\right) = \sum_{i=1}^{s} \frac{\lambda_i}{1+\lambda_i}$$
$$\operatorname{tr}\left(W^{-1}B\right) = \sum_{i=1}^{s} \lambda_i.$$

6. Using the same notation as Problem 5, show that
   (a) $\operatorname{tr} B(W+B)^{-1} \le -\log \frac{|W|}{|W+B|} \le \operatorname{tr} BW^{-1}$;
   (b) $\sum_{i=1}^{s} \frac{\lambda_i}{1+\lambda_i} \le \log \prod_{i=1}^{s} (1+\lambda_i) \le \sum_{i=1}^{s} \lambda_i$.

[1] This is called the *multivariate analysis of covariance*.